

# Orbit Approach to Separation of Variables in $\mathfrak{sl}(4)$ -Related Integrable Systems

Julia Bernatska

National University of 'Kyiv Mohyla Academy'  
BernatskaJM@ukma.kiev.ua

February 28, 2014

## Abstract

Separation of variables by means of the orbit method is implemented to integrable systems on coadjoint orbits in an  $\mathfrak{sl}(4)$  loop algebra. This is a development and a kind of explanation for Sklyanin's procedure of separation of variables. It is shown that points on a spectral curve serve as variables of separation for two integrable systems living on two generic orbits embedded into a common manifold. These orbits are endowed with different nonsingular Lie-Poisson brackets. Explicit expressions for the case of  $\mathfrak{sl}(4)$  loop algebra are given.

## 1 Introduction

Here we continue to develop the orbit approach to separation of variables presented in [1], where  $\mathfrak{sl}(3)$ -related integrable systems are considered. Separation of variables in dynamical systems on coadjoint orbits in an  $\mathfrak{sl}(3)$  loop algebra are discussed in a number of papers, cited in [1]. Now we consider the case of  $\mathfrak{sl}(4)$  loop algebra, uninvestigated yet because of its computational complexity. We try to overcome this complexity by means of the orbit approach.

## 2 Preliminaries

First we construct a loop algebra based on the algebra  $\mathfrak{g} = \mathfrak{sl}(4, \mathbb{C})$  with the Cartan-Weyl basis

$$\begin{aligned} X_1 &= E_{12}, & X_2 &= E_{23}, & X_3 &= E_{13}, & X_4 &= E_{34}, & X_5 &= E_{24}, & X_6 &= E_{14}, \\ Y_1 &= E_{21}, & Y_2 &= E_{32}, & Y_3 &= E_{31}, & Y_4 &= E_{43}, & Y_5 &= E_{42}, & Y_6 &= E_{41}, \\ H_1 &= \frac{1}{4} \text{diag}(3, -1, -1, -1), & H_2 &= \frac{1}{2} \text{diag}(1, 1, -1, -1), & H_3 &= \frac{1}{4} \text{diag}(1, 1, 1, -3). \end{aligned}$$

By  $E_{ij}$  we denote the standard basis in the vector space of  $4 \times 4$  matrices, i. e.  $E_{ij}$  is the matrix with only 1 at the position of row  $i$  column  $j$  and 0s at all other positions. The matrices  $H_1, H_2, H_3$  are chosen to be the dual basis to the standard basis in the Cartan subalgebra:

$$H_1^* = \text{diag}(1, -1, 0, 0), \quad H_2^* = \text{diag}(0, 1, -1, 0), \quad H_3^* = \text{diag}(0, 0, 1, -1)$$

with respect to the bilinear form  $\langle A, B \rangle = \text{Tr } AB$ . In what follows we denote the set  $\{H_1, Y_1, X_1, Y_3, X_3, Y_6, X_6, H_2, Y_2, X_2, Y_5, X_5, H_3, Y_4, X_4\}$  by  $\{Z_a\}_{a=1}^{15}$ . We also introduce the dual algebra  $\mathfrak{g}^*$  with respect to the mentioned bilinear form, we denote its basis by  $\{Z_a^*\}$ :

$$X_j^* = Y_j, \quad Y_j^* = X_j, \quad j = 1, \dots, 6, \quad H_1^*, \quad H_2^*, \quad H_3^*.$$

Let  $\mathcal{P}(\lambda, \lambda^{-1})$  be the algebra of Laurent polynomials in  $\lambda$ , and  $\tilde{\mathfrak{g}}$  be the loop algebra  $\mathfrak{sl}(4, \mathbb{C}) \otimes \mathcal{P}(\lambda, \lambda^{-1})$ . Then

$$Z_a^m = \lambda^m Z_a, \quad m \in \mathbb{Z}, \quad a = 1, \dots, 15$$

form a basis in  $\tilde{\mathfrak{g}}$ . This loop algebra is standard graded with respect to the operator  $\mathfrak{d} = d/d\lambda$  of homogeneous degree. By  $\mathfrak{g}_m$ ,  $m \in \mathbb{Z}$  we denote the eigenspace of degree  $m$ .

According to the Kostant-Adler scheme [2]  $\tilde{\mathfrak{g}}$  is decomposed into two subalgebras

$$\tilde{\mathfrak{g}}_+ = \sum_{m \geq 0} \mathfrak{g}_m, \quad \tilde{\mathfrak{g}}_- = \sum_{m < 0} \mathfrak{g}_m, \quad \tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_+ + \tilde{\mathfrak{g}}_-.$$

Further, we introduce the ad-invariant bilinear forms

$$\langle A(\lambda), B(\lambda) \rangle_k = \operatorname{res}_{\lambda=0} \lambda^{-k-1} \operatorname{Tr} A(\lambda) B(\lambda), \quad A(\lambda), B(\lambda) \in \tilde{\mathfrak{g}}, \quad k \in \mathbb{Z}$$

and use them to define the spaces dual to  $\tilde{\mathfrak{g}}_+$  and  $\tilde{\mathfrak{g}}_-$ .

**Example 1.** With respect to the bilinear form  $\langle A(\lambda), B(\lambda) \rangle_{-1} = \operatorname{res}_{\lambda=0} \operatorname{Tr} A(\lambda) B(\lambda)$  we obtain the following dual spaces

$$(\tilde{\mathfrak{g}}_-)^* = \tilde{\mathfrak{g}}_+, \quad (\tilde{\mathfrak{g}}_+)^* = \tilde{\mathfrak{g}}_-,$$

where  $(\tilde{\mathfrak{g}}_-)^*$  and  $(\tilde{\mathfrak{g}}_+)^*$  contain only nonzero functionals on  $\tilde{\mathfrak{g}}_{\pm}$ .

**Example 2.** With respect to the bilinear form  $\langle A(\lambda), B(\lambda) \rangle_{N-1} = \lambda^{-N} \operatorname{res}_{\lambda=0} \operatorname{Tr} A(\lambda) B(\lambda)$  we obtain the dual spaces

$$(\tilde{\mathfrak{g}}_-)^* = \sum_{m \geq N} \mathfrak{g}_m, \quad (\tilde{\mathfrak{g}}_+)^* = \sum_{m < N} \mathfrak{g}_m.$$

Further we consider a subset of  $\tilde{\mathfrak{g}}_+$  as a dual space to the both of subalgebras:  $\tilde{\mathfrak{g}}_+$  and  $\tilde{\mathfrak{g}}_-$ . We are interested in its foliations into orbits of the coadjoint action of these two subalgebras.

### 3 Orbits of $\mathfrak{sl}(4, \mathbb{C}) \otimes \mathcal{P}(\lambda, \lambda^{-1})$ as phase spaces

Fixing  $N \geq 0$  we introduce the variables  $\{\alpha_1^{(m)}, \beta_1^{(m)}, \gamma_1^{(m)}, \beta_3^{(m)}, \gamma_3^{(m)}, \beta_6^{(m)}, \gamma_6^{(m)}, \alpha_2^{(m)}, \beta_2^{(m)}, \gamma_2^{(m)}, \beta_5^{(m)}, \gamma_5^{(m)}, \alpha_3^{(m)}, \beta_4^{(m)}, \gamma_4^{(m)} : m = 0, 1, \dots, N\}$  denoted all together by  $\{L_a^{(m)}\}_{a=1}^{15}$ . Consider a space  $\mathcal{M} \in \tilde{\mathfrak{g}}^*$  of the elements

$$\mathbf{L}(\lambda) = \sum_{m=0}^N \sum_{a=1}^{\operatorname{rank} \mathfrak{g}} L_a^{(m)} (Z_a^m)^* = \begin{pmatrix} \alpha_1(\lambda) & \beta_1(\lambda) & \beta_3(\lambda) & \beta_6(\lambda) \\ \gamma_1(\lambda) & \alpha_2(\lambda) - \alpha_1(\lambda) & \beta_2(\lambda) & \beta_5(\lambda) \\ \gamma_3(\lambda) & \gamma_2(\lambda) & \alpha_3(\lambda) - \alpha_2(\lambda) & \beta_4(\lambda) \\ \gamma_6(\lambda) & \gamma_5(\lambda) & \gamma_4(\lambda) & -\alpha_3(\lambda) \end{pmatrix}, \quad (1)$$

where

$$L_a(\lambda) = \sum_{m=0}^N \lambda^m L_a^{(m)}.$$

Let  $\mathcal{C}(\mathcal{M})$  be the space of smooth functions on  $\mathcal{M}$ . For all  $f_1, f_2 \in \mathcal{C}(\mathcal{M})$  we define the first Lie-Poisson bracket by the formula

$$\{f_1, f_2\}_{\mathfrak{f}} = \sum_{m,n=0}^N \sum_{a,b=1}^{\operatorname{rank} \mathfrak{g}} P_{ab}^{mn}(-1) \frac{\partial f_1}{\partial L_a^{(m)}} \frac{\partial f_2}{\partial L_b^{(n)}}, \quad (2)$$

$$P_{ab}^{mn}(-1) = \langle \mathbf{L}(\lambda), [Z_a^{-m-1}, Z_b^{-n-1}] \rangle_{-1},$$

and turn the space  $\mathcal{C}(\mathcal{M})$  into a phase space denoted by  $\mathcal{D}_{\mathfrak{f}}$ . The variables  $\{L_a^{(N)}\}$  annihilate the bracket (2), therefore we consider them as constant in  $\mathcal{D}_{\mathfrak{f}}$ . To make the introduced bracket nonsingular we restrict it to the subspace  $\mathcal{M}_0$  of  $\mathcal{M}$  defined by the constraints

$$L_a^{(N)} = \text{const}, \quad a = 1, \dots, 15.$$

For all  $f_1, f_2 \in \mathcal{C}(\mathcal{M}_0)$  we define the second Lie-Poisson bracket by the formula

$$\{f_1, f_2\}_{\mathfrak{s}} = \sum_{m,n=0}^N \sum_{a,b=1}^{\operatorname{rank} \mathfrak{g}} P_{ab}^{mn}(N-1) \frac{\partial f_1}{\partial L_a^{(m)}} \frac{\partial f_2}{\partial L_b^{(n)}}, \quad (3)$$

$$P_{ab}^{mn}(N-1) = \langle L(\lambda), [Z_a^{-m+N-1}, Z_b^{-n+N-1}] \rangle_{N-1}$$

and introduce another phase space denoted by  $\mathcal{D}_s$ . In what follows we consider the space of smooth functions  $\mathcal{C}(\mathcal{M}_0)$ , and use the set  $\{L_a^{(m)} \mid m=1, \dots, N-1\}$  as *dynamic variables* in it. We call  $\mathcal{M}_0$  a *finite gap sector of  $\tilde{\mathfrak{g}}$* , more precisely the  $N$ -gap sector.

**Remark 1.** In addition to the brackets (2) and (3), one can define intermediate brackets with the Poisson tensors

$$P_{ab}^{mn}(k) = \langle L(\lambda), [Z_a^{-m+k}, Z_b^{-n+k}] \rangle_k, \quad k = 0, \dots, N-2. \quad (4)$$

According to Example 1 we consider  $\mathcal{M}_0$  as located in  $(\tilde{\mathfrak{g}}_-)^*$  with respect to the bilinear form  $\langle \cdot, \cdot \rangle_{-1}$ . Obviously,  $\mathcal{M}_0$  is  $\text{ad}^*$ -invariant under the coadjoint action of the factor-algebra  $\tilde{\mathfrak{g}}_- / \sum_{l < -N} \mathfrak{g}_l$ . In this connection we introduce the first Lie-Poisson bracket  $\{\cdot, \cdot\}_f$ .

On the other hand, according to Example 2 we consider  $\mathcal{M}_0$  as located in  $(\tilde{\mathfrak{g}}_+)^*$  with respect to the bilinear form  $\langle \cdot, \cdot \rangle_{N-1}$ . One can see that  $\mathcal{M}_0$  is  $\text{ad}^*$ -invariant also under the coadjoint action of the factor-algebra  $\tilde{\mathfrak{g}}_+ / \sum_{l \geq N} \mathfrak{g}_l$ . So we introduce the second Lie-Poisson bracket  $\{\cdot, \cdot\}_s$ .

Next, we introduce the following  $\text{ad}^*$ -invariant functions in  $\lambda$  (for the sake of simplicity we will often omit writing the dependence on  $\lambda$ )

$$\begin{aligned} I_2(\lambda) &\equiv \frac{1}{2} \text{Tr } L^2(\lambda) = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 - \alpha_1\alpha_2 - \alpha_2\alpha_3 + \beta_1\gamma_1 + \beta_2\gamma_2 + \beta_3\gamma_3 + \beta_4\gamma_4 + \beta_5\gamma_5 + \beta_6\gamma_6 = \\ &= - \begin{vmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \alpha_2 - \alpha_1 \end{vmatrix} - \begin{vmatrix} \alpha_2 - \alpha_1 & \beta_2 \\ \gamma_2 & \alpha_3 - \alpha_2 \end{vmatrix} - \begin{vmatrix} \alpha_1 & \beta_3 \\ \gamma_3 & \alpha_3 - \alpha_2 \end{vmatrix} - \begin{vmatrix} \alpha_3 - \alpha_2 & \beta_4 \\ \gamma_4 & -\alpha_3 \end{vmatrix} - \\ &\quad - \begin{vmatrix} \alpha_2 - \alpha_1 & \beta_5 \\ \gamma_5 & -\alpha_3 \end{vmatrix} - \begin{vmatrix} \alpha_1 & \beta_6 \\ \gamma_6 & -\alpha_3 \end{vmatrix}, \end{aligned} \quad (5)$$

$$\begin{aligned} I_3(\lambda) &\equiv \frac{1}{3} \text{Tr } L^3(\lambda) = \alpha_2\alpha_1^2 - \alpha_1\alpha_2^2 + \alpha_3\alpha_2^2 - \alpha_2\alpha_3^2 + \beta_1\gamma_1\alpha_2 + \beta_2\gamma_2(\alpha_3 - \alpha_1) + \beta_3\gamma_3(\alpha_1 - \alpha_2 + \alpha_3) + \\ &\quad + \beta_3\gamma_1\gamma_2 + \beta_1\beta_2\gamma_3 + \beta_4[\beta_2\gamma_5 + \beta_3\gamma_6 - \alpha_2\gamma_4] + \beta_5[\gamma_2\gamma_4 + \beta_1\gamma_6 - (\alpha_1 - \alpha_2 + \alpha_3)\gamma_5] + \\ &\quad + \beta_6[\gamma_3\gamma_4 + \gamma_1\gamma_5 - (\alpha_3 - \alpha_1)\gamma_6] = \begin{vmatrix} \alpha_1 & \beta_1 & \beta_3 \\ \gamma_1 & \alpha_2 - \alpha_1 & \beta_2 \\ \gamma_3 & \gamma_2 & \alpha_3 - \alpha_2 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \beta_1 & \beta_6 \\ \gamma_1 & \alpha_2 - \alpha_1 & \beta_5 \\ \gamma_6 & \gamma_5 & -\alpha_3 \end{vmatrix} + \\ &\quad + \begin{vmatrix} \alpha_1 & \beta_3 & \beta_6 \\ \gamma_3 & \alpha_3 - \alpha_2 & \beta_4 \\ \gamma_6 & \gamma_4 & -\alpha_3 \end{vmatrix} + \begin{vmatrix} \alpha_2 - \alpha_1 & \beta_2 & \beta_5 \\ \gamma_2 & \alpha_3 - \alpha_2 & \beta_4 \\ \gamma_5 & \gamma_4 & -\alpha_3 \end{vmatrix}, \end{aligned}$$

$$\begin{aligned} I_4(\lambda) &\equiv \frac{1}{4} [\text{Tr } L^4(\lambda) - \frac{1}{2} (\text{Tr } L^2(\lambda))^2] = \alpha_3 \begin{vmatrix} \alpha_1 & \beta_1 & \beta_3 \\ \gamma_1 & \alpha_2 - \alpha_1 & \beta_2 \\ \gamma_3 & \gamma_2 & \alpha_3 - \alpha_2 \end{vmatrix} + \beta_4 \begin{vmatrix} \alpha_1 & \beta_1 & \beta_3 \\ \gamma_1 & \alpha_2 - \alpha_1 & \beta_2 \\ \gamma_6 & \gamma_5 & \gamma_4 \end{vmatrix} - \\ &\quad - \beta_5 \begin{vmatrix} \alpha_1 & \beta_1 & \beta_3 \\ \gamma_3 & \gamma_2 & \alpha_3 - \alpha_2 \end{vmatrix} + \beta_6 \begin{vmatrix} \gamma_1 & \alpha_2 - \alpha_1 & \beta_2 \\ \gamma_3 & \gamma_2 & \alpha_3 - \alpha_2 \\ \gamma_6 & \gamma_5 & \gamma_4 \end{vmatrix} = -\det L(\lambda). \end{aligned} \quad (6)$$

Every function  $I_k$  is a sum of the diagonal minors of order  $k$  up to a sign. The functions  $\{I_1=0, I_2, \dots, I_{\text{rank } \mathfrak{g}}\}$  serve as coefficients of the characteristic polynomial of the  $L$ -matrix, in our case:

$$\chi(w) = w^4 - I_2 w^2 - I_3 w - I_4.$$

The functions  $I_2, I_3, I_4$  are polynomials in the spectral parameter  $\lambda$ , and their coefficients serve as invariant functions in dynamic variables, namely:

$$I_2(\lambda) = h_2^{(0)} + h_2^{(1)}\lambda + \dots + h_2^{(2N)}\lambda^{2N}, \quad I_3(\lambda) = h_3^{(0)} + h_3^{(1)}\lambda + \dots + h_3^{(3N)}\lambda^{3N}, \quad (7)$$

$$I_4(\lambda) = h_4^{(0)} + h_4^{(1)}\lambda + \dots + h_4^{(4N)}\lambda^{4N},$$

$$\begin{aligned} h_2^{(\nu)} &= - \sum_{m+n=\nu} \left( \begin{vmatrix} \alpha_1^{(m)} & \beta_1^{(n)} \\ \gamma_1^{(m)} & \alpha_2^{(n)} - \alpha_1^{(n)} \end{vmatrix} + \begin{vmatrix} \alpha_2^{(m)} - \alpha_1^{(m)} & \beta_2^{(n)} \\ \gamma_2^{(m)} & \alpha_3^{(n)} - \alpha_2^{(n)} \end{vmatrix} + \begin{vmatrix} \alpha_1^{(m)} & \beta_3^{(n)} \\ \gamma_3^{(m)} & \alpha_3^{(n)} - \alpha_2^{(n)} \end{vmatrix} - \right. \\ &\quad \left. - \begin{vmatrix} \alpha_3^{(m)} - \alpha_2^{(m)} & \beta_4^{(n)} \\ \gamma_4^{(m)} & -\alpha_3^{(n)} \end{vmatrix} - \begin{vmatrix} \alpha_2^{(m)} - \alpha_1^{(m)} & \beta_5^{(n)} \\ \gamma_5^{(m)} & -\alpha_3^{(n)} \end{vmatrix} - \begin{vmatrix} \alpha_1^{(m)} & \beta_6^{(n)} \\ \gamma_6^{(m)} & -\alpha_3^{(n)} \end{vmatrix} \right), \\ \nu &= 0, 1, \dots, 2N; \end{aligned} \quad (8)$$

$$\begin{aligned}
h_3^{(\nu)} &= \sum_{m+n+k=\nu} \left( \begin{vmatrix} \alpha_1^{(m)} & \beta_1^{(n)} & \beta_3^{(k)} \\ \gamma_1^{(m)} & \alpha_2^{(n)} - \alpha_1^{(n)} & \beta_2^{(k)} \\ \gamma_3^{(m)} & \gamma_2^{(n)} & \alpha_3^{(k)} - \alpha_2^{(k)} \end{vmatrix} + \begin{vmatrix} \alpha_1^{(m)} & \beta_1^{(n)} & \beta_6^{(k)} \\ \gamma_1^{(m)} & \alpha_2^{(n)} - \alpha_1^{(n)} & \beta_5^{(k)} \\ \gamma_6^{(m)} & \gamma_5^{(n)} & -\alpha_3^{(k)} \end{vmatrix} + \right. \\
&\quad \left. + \begin{vmatrix} \alpha_1^{(m)} & \beta_3^{(n)} & \beta_6^{(k)} \\ \gamma_3^{(m)} & \alpha_3^{(n)} - \alpha_2^{(n)} & \beta_4^{(k)} \\ \gamma_6^{(m)} & \gamma_4^{(n)} & -\alpha_3^{(k)} \end{vmatrix} + \begin{vmatrix} \alpha_2^{(m)} - \alpha_1^{(m)} & \beta_2^{(n)} & \beta_5^{(k)} \\ \gamma_2^{(m)} & \alpha_3^{(n)} - \alpha_2^{(n)} & \beta_4^{(k)} \\ \gamma_5^{(m)} & \gamma_4^{(n)} & -\alpha_3^{(k)} \end{vmatrix} \right), \quad \nu = 0, 1, \dots, 3N, \\
h_4^{(\nu)} &= - \sum_{\substack{m+n+ \\ +k+j=\nu}} \begin{vmatrix} \alpha_1^{(m)} & \beta_1^{(n)} & \beta_3^{(k)} & \beta_6^{(j)} \\ \gamma_1^{(m)} & \alpha_2^{(n)} - \alpha_1^{(n)} & \beta_2^{(k)} & \beta_5^{(j)} \\ \gamma_3^{(m)} & \gamma_2^{(n)} & \alpha_3^{(k)} - \alpha_2^{(k)} & \beta_4^{(j)} \\ \gamma_6^{(m)} & \gamma_5^{(n)} & \beta_4^{(k)} & -\alpha_3^{(j)} \end{vmatrix}, \quad \nu = 0, 1, \dots, 4N.
\end{aligned}$$

Evidently,  $h_2^{(2N)}$ ,  $h_3^{(3N)}$ ,  $h_4^{(4N)}$  are constant, for they do not contain dynamic variables.

The following assertions are immediately derived from the Kostant-Adler scheme [2].

**Proposition 3.1.** *All functions  $h_2^{(0)}$ ,  $h_2^{(1)}, \dots, h_2^{(2N-1)}$ ,  $h_3^{(0)}$ ,  $h_3^{(1)}, \dots, h_3^{(3N-1)}$ ,  $h_4^{(0)}$ ,  $h_4^{(1)}, \dots, h_4^{(4N-1)}$  defined by (8) are in involution with respect to the brackets (2) and (3).*

**Proposition 3.2.** *The functions  $h_2^{(\nu)}$ ,  $h_3^{(\nu+N)}$ ,  $h_4^{(\nu+2N)}$ ,  $\nu = N, \dots, 2N-1$  are functionally independent on  $\mathcal{M}_0$  and annihilate the first Lie-Poisson bracket (2).*

Let  $\mathcal{O}_f \subset \mathcal{M}_0$  be the algebraic manifold defined by

$$h_2^{(\nu)} = c_2^{(\nu)}, \quad h_3^{(\nu+N)} = c_3^{(\nu+N)}, \quad h_4^{(\nu+2N)} = c_4^{(\nu+2N)}, \quad \nu = N, \dots, 2N-1, \quad (9)$$

where all  $c_2^{(\nu)}$ ,  $c_3^{(\nu+N)}$ ,  $c_4^{(\nu+2N)}$  are fixed complex numbers. The manifold  $\mathcal{O}_f$  is a generic orbit of coadjoint action of the subalgebra  $\tilde{\mathfrak{g}}_-$  on  $\mathcal{M}_0$ ,  $\dim \mathcal{O}_f = 12N$ . Variation of the constants  $c_2^{(\nu)}$ ,  $c_3^{(\nu+N)}$ ,  $c_4^{(\nu+2N)}$  gives a foliation of  $\mathcal{M}_0$  into orbits of the first type. Every orbit serves as a symplectic leaf in the symplectic manifold  $\mathcal{M}_0$ .

**Proposition 3.3.** *The functions  $h_2^{(\nu)}$ ,  $h_3^{(\nu)}$ ,  $h_4^{(\nu)}$ ,  $\nu = 0, \dots, N-1$  are functionally independent on  $\mathcal{M}_0$  and annihilate the second Lie-Poisson bracket (3).*

The algebraic manifold  $\mathcal{O}_s \subset \mathcal{M}_0$  defined by

$$h_2^{(\nu)} = c_2^{(\nu)}, \quad h_3^{(\nu)} = c_3^{(\nu)}, \quad h_4^{(\nu)} = c_4^{(\nu)}, \quad \nu = 0, \dots, N-1, \quad (10)$$

where all  $c_2^{(\nu)}$ ,  $c_3^{(\nu)}$ ,  $c_4^{(\nu)}$  are fixed complex numbers, is a generic orbit of coadjoint action of the subalgebra  $\tilde{\mathfrak{g}}_+$  on  $\mathcal{M}_0$ ,  $\dim \mathcal{O}_s = 12N$ . Variation of the constants  $c_2^{(\nu)}$ ,  $c_3^{(\nu)}$ ,  $c_4^{(\nu)}$  gives another foliation of  $\mathcal{M}_0$  into orbits of the second type. In what follows we call  $\mathcal{O}_f$  and  $\mathcal{O}_s$  simply orbits, and call (9), (10) orbit equations.

## 4 Separation of variables on $\mathcal{O}_f$

The orbit  $\mathcal{O}_f$  with the first Lie-Poisson bracket (2) has the following Poisson structure:

$$\{L_{ij}^{(m)}, L_{kl}^{(n)}\}_f = L_{kj}^{(m+n+1)} \delta_{il} - L_{il}^{(m+n+1)} \delta_{kj}, \quad (11)$$

that can be written in terms of the  $\mathbf{r}$ -matrix

$$\begin{aligned}
\{L_1(u) \otimes L_2(v)\}_f &= [r_{12}(u-v), L_1(u) + L_2(v)] \\
r_{12}(u-v) &= \frac{1}{u-v} \sum_{a,b} \langle Z_a, Z_b \rangle Z_a^* \otimes Z_b^*
\end{aligned} \quad (12)$$

with  $L_1(u) = L(u) \otimes \mathbb{I}$ ,  $L_2(v) = \mathbb{I} \otimes L(v)$ , where  $\mathbb{I}$  is the identity matrix.

In order to parameterize the orbit  $\mathcal{O}_f$  of dimension  $12N$  we need to eliminate  $3N$  variables among  $15N$  variables  $\{L_a^{(m)}\}$ . It is convenient to eliminate such set of variables that all the invariant functions are linear in it. Such set of dynamic variables correspond to one of the maximal sets of nilpotent commuting basis elements. Here we choose the variables  $\{\beta_4^{(m)}, \beta_5^{(m)}, \beta_6^{(m)}\}$  for elimination.

Using linearity of the orbit equations (9) in the chosen set of variables one can write them in the matrix form

$$\mathbf{c} = \mathbf{F}^+ \boldsymbol{\beta} + \mathbf{a}^+, \quad (13a)$$

$$\mathbf{F}^+ = \begin{bmatrix} \mathbf{F}_N & \mathbf{F}_{N-1} & \dots & \mathbf{F}_1 & \mathbf{F}_0 \\ 0 & \mathbf{F}_N & \dots & \mathbf{F}_2 & \mathbf{F}_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \mathbf{F}_N & \mathbf{F}_{N-1} \\ 0 & 0 & \dots & 0 & \mathbf{F}_N \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \beta^{(0)} \\ \beta^{(1)} \\ \vdots \\ \beta^{(N-1)} \\ \beta^{(N)} \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} \mathbf{c}_N \\ \mathbf{c}_{N+1} \\ \vdots \\ \mathbf{c}_{2N-1} \\ \mathbf{c}_{2N} \end{bmatrix}, \quad \mathbf{a}^+ = \begin{bmatrix} \mathbf{a}_N \\ \mathbf{a}_{N+1} \\ \vdots \\ \mathbf{a}_{2N-1} \\ \mathbf{a}_{2N} \end{bmatrix},$$

$$\beta^{(j)} = \begin{bmatrix} \beta_6^{(j)} \\ \beta_5^{(j)} \\ \beta_4^{(j)} \end{bmatrix}, \quad \mathbf{c}_j = \begin{bmatrix} c_2^{(j)} \\ c_3^{(j+N)} \\ c_4^{(j+2N)} \end{bmatrix}, \quad \mathbf{a}_j = \begin{bmatrix} a_2^{(j)} \\ a_3^{(j+N)} \\ a_4^{(j+2N)} \end{bmatrix}, \quad \mathbf{F}_j = \begin{bmatrix} B_{16}^{(j)} & B_{15}^{(j)} & B_{14}^{(j)} \\ B_{26}^{(j+N)} & B_{25}^{(j+N)} & B_{24}^{(j+N)} \\ B_{36}^{(j+2N)} & B_{35}^{(j+2N)} & B_{34}^{(j+2N)} \end{bmatrix},$$

$$B_{16}^{(j)} = \gamma_6^{(j)}, \quad B_{15}^{(j)} = \gamma_5^{(j)}, \quad B_{14}^{(j)} = \gamma_4^{(j)},$$

$$B_{26}^{(j)} = \sum_{\substack{m+n+j \\ +n=j}} \left( \left| \begin{matrix} \gamma_1^{(m)} & \alpha_2^{(n)} - \alpha_1^{(n)} \\ \gamma_6^{(m)} & \gamma_5^{(n)} \end{matrix} \right| + \left| \begin{matrix} \gamma_3^{(m)} & \alpha_3^{(n)} - \alpha_2^{(n)} \\ \gamma_6^{(m)} & \gamma_4^{(n)} \end{matrix} \right| \right), \quad B_{36}^{(j)} = \sum_{\substack{m+n+j \\ +k=j}} \left| \begin{matrix} \gamma_1^{(m)} & \alpha_2^{(n)} - \alpha_1^{(n)} & \beta_2^{(k)} \\ \gamma_3^{(m)} & \gamma_2^{(n)} & \alpha_3^{(k)} - \alpha_2^{(k)} \\ \gamma_6^{(m)} & \gamma_5^{(n)} & \gamma_4^{(k)} \end{matrix} \right|,$$

$$B_{25}^{(j)} = \sum_{\substack{m+n+j \\ +n=j}} \left( - \left| \begin{matrix} \alpha_1^{(m)} & \beta_1^{(n)} \\ \gamma_6^{(m)} & \gamma_5^{(n)} \end{matrix} \right| + \left| \begin{matrix} \gamma_2^{(m)} & \alpha_3^{(n)} - \alpha_2^{(n)} \\ \gamma_5^{(m)} & \gamma_4^{(n)} \end{matrix} \right| \right), \quad B_{35}^{(j)} = - \sum_{\substack{m+n+j \\ +k=j}} \left| \begin{matrix} \alpha_1^{(m)} & \beta_1^{(n)} & \beta_3^{(k)} \\ \gamma_3^{(m)} & \gamma_2^{(n)} & \alpha_3^{(k)} - \alpha_2^{(k)} \\ \gamma_6^{(m)} & \gamma_5^{(n)} & \gamma_4^{(k)} \end{matrix} \right|, \quad (13b)$$

$$B_{24}^{(j)} = \sum_{\substack{m+n+j \\ +n=j}} \left( - \left| \begin{matrix} \alpha_1^{(m)} & \beta_3^{(n)} \\ \gamma_6^{(m)} & \gamma_4^{(n)} \end{matrix} \right| - \left| \begin{matrix} \alpha_2^{(m)} - \alpha_1^{(m)} & \beta_2^{(n)} \\ \gamma_5^{(m)} & \gamma_4^{(n)} \end{matrix} \right| \right), \quad B_{34}^{(j)} = \sum_{\substack{m+n+j \\ +k=j}} \left| \begin{matrix} \alpha_1^{(m)} & \beta_1^{(n)} & \beta_3^{(k)} \\ \gamma_1^{(m)} & \alpha_2^{(n)} - \alpha_1^{(n)} & \beta_2^{(k)} \\ \gamma_6^{(m)} & \gamma_5^{(n)} & \gamma_4^{(k)} \end{matrix} \right|,$$

$$A_2^{(j)} = \sum_{\substack{m+n+j \\ +n=j}} \left( - \left| \begin{matrix} \alpha_1^{(m)} & \beta_1^{(n)} \\ \gamma_1^{(m)} & \alpha_2^{(n)} - \alpha_1^{(n)} \end{matrix} \right| - \left| \begin{matrix} \alpha_2^{(m)} - \alpha_1^{(m)} & \beta_2^{(n)} \\ \gamma_2^{(m)} & \alpha_3^{(n)} - \alpha_2^{(n)} \end{matrix} \right| - \left| \begin{matrix} \alpha_1^{(m)} & \beta_3^{(n)} \\ \gamma_3^{(m)} & \alpha_3^{(n)} - \alpha_2^{(n)} \end{matrix} \right| \right),$$

$$A_3^{(j)} = \sum_{\substack{m+n+j \\ +k=j}} \left| \begin{matrix} \alpha_1^{(m)} & \beta_1^{(n)} & \beta_3^{(k)} \\ \gamma_1^{(m)} & \alpha_2^{(n)} - \alpha_1^{(n)} & \beta_2^{(k)} \\ \gamma_3^{(m)} & \gamma_2^{(n)} & \alpha_3^{(k)} - \alpha_2^{(k)} \end{matrix} \right|, \quad a_2^{(j)} = A_2^{(j)} + \sum_{m+n=j} \alpha_3^{(m)} \alpha_3^{(n)}, \quad (13c)$$

$$a_3^{(j)} = A_3^{(j)} + \sum_{m+n=j} \alpha_3^{(m)} A_2^{(n)}, \quad a_4^{(j)} = \sum_{m+n=j} \alpha_3^{(m)} A_3^{(n)}.$$

Supposing  $\mathbf{F}_N$  is nonsingular, one easily eliminates the variables  $\boldsymbol{\beta}$

$$\boldsymbol{\beta} = (\mathbf{F}^+)^{-1}(\mathbf{c} - \mathbf{a}^+), \quad \text{or}$$

$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{N-1} \\ \beta_N \end{bmatrix} = \begin{bmatrix} \mathbf{F}_N^{-1} & \tilde{\mathbf{F}}_{N-1} & \dots & \tilde{\mathbf{F}}_1 & \tilde{\mathbf{F}}_0 \\ 0 & \mathbf{F}_N^{-1} & \dots & \tilde{\mathbf{F}}_2 & \tilde{\mathbf{F}}_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \mathbf{F}_N^{-1} & \tilde{\mathbf{F}}_{N-1} \\ 0 & 0 & \dots & 0 & \mathbf{F}_N^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{c}_N - \mathbf{a}_N \\ \mathbf{c}_{N+1} - \mathbf{a}_{N+1} \\ \vdots \\ \mathbf{c}_{2N-1} - \mathbf{a}_{2N-1} \\ \mathbf{c}_{2N} - \mathbf{a}_{2N} \end{bmatrix},$$

$$\tilde{\mathbf{F}}_{N-n} = \mathbf{F}_N^{-1} \sum_{k=1}^n (-\mathbf{F}_{N-n-1+k} \mathbf{F}_N^{-1})^k, \quad n = 1, \dots, N.$$

Next, substitute  $\boldsymbol{\beta}$  into the Hamiltonians  $h_2^{(0)}, h_2^{(1)}, \dots, h_2^{(N-1)}, h_3^{(0)}, h_3^{(1)}, \dots, h_3^{(2N-1)}, h_4^{(0)}, h_4^{(1)}, \dots, h_4^{(3N-1)}$

$$\mathbf{h} = \mathbf{F}^- \boldsymbol{\beta} + \mathbf{a}^- = \mathbf{F}^- (\mathbf{F}^+)^{-1} \mathbf{c} + \mathbf{a}^- - \mathbf{F}^- (\mathbf{F}^+)^{-1} \mathbf{a}^+, \quad (14)$$

where

$$\begin{aligned}
F^- &= \begin{bmatrix} \mathbf{g}_0 & 0 & \cdots & 0 & 0 \\ \mathbf{g}_1 & \mathbf{g}_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{g}_{N-1} & \mathbf{g}_{N-2} & \cdots & \mathbf{g}_0 & 0 \\ \mathbf{G}_0 & \mathbf{g}_{N-1}^0 & \cdots & \mathbf{g}_1^0 & \mathbf{g}_0^0 \\ \mathbf{G}_1 & \mathbf{G}_0 & \cdots & \mathbf{g}_2^0 & \mathbf{g}_1^0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{G}_{N-1} & \mathbf{G}_{N-2} & \cdots & \mathbf{g}_0^0 & \mathbf{g}_{N-1}^0 \\ \mathbf{F}_0 & \mathbf{G}_{N-1}^0 & \cdots & \mathbf{G}_1^0 & \mathbf{G}_0^0 \\ \mathbf{F}_1 & \mathbf{F}_0 & \cdots & \mathbf{G}_2^0 & \mathbf{G}_1^0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{F}_{N-1} & \mathbf{F}_{N-2} & \cdots & \mathbf{F}_0 & \mathbf{G}_{N-1}^0 \end{bmatrix}, \quad \mathbf{h} = \begin{bmatrix} h_4^{(0)} \\ h_4^{(1)} \\ \vdots \\ h_4^{(N-1)} \\ \check{\mathbf{h}}_0 \\ \check{\mathbf{h}}_1 \\ \vdots \\ \check{\mathbf{h}}_{N-1} \\ \mathbf{h}_0 \\ \mathbf{h}_1 \\ \vdots \\ \mathbf{h}_{N-1} \end{bmatrix}, \quad \mathbf{a}^- = \begin{bmatrix} a_4^{(0)} \\ a_4^{(1)} \\ \vdots \\ a_4^{(N-1)} \\ \check{\mathbf{a}}_0 \\ \check{\mathbf{a}}_1 \\ \vdots \\ \check{\mathbf{a}}_{N-1} \\ \mathbf{a}_0 \\ \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_{N-1} \end{bmatrix}, \\
\mathbf{g}_j &= \begin{bmatrix} B_{36}^{(j)} & B_{35}^{(j)} & B_{34}^{(j)} \end{bmatrix}, \quad \mathbf{G}_j = \begin{bmatrix} B_{26}^{(j)} & B_{25}^{(j)} & B_{24}^{(j)} \\ B_{36}^{(j+N)} & B_{35}^{(j+N)} & B_{34}^{(j+N)} \end{bmatrix}, \\
\mathbf{g}_j^0 &= \begin{bmatrix} 0 & 0 & 0 \\ B_{36}^{(j)} & B_{35}^{(j)} & B_{34}^{(j)} \end{bmatrix}, \quad \mathbf{G}_j^0 = \begin{bmatrix} 0 & 0 & 0 \\ B_{26}^{(j)} & B_{25}^{(j)} & B_{24}^{(j)} \\ B_{36}^{(j+N)} & B_{35}^{(j+N)} & B_{34}^{(j+N)} \end{bmatrix}, \\
\check{\mathbf{a}}_j &= \begin{bmatrix} a_3^{(j)} \\ a_4^{(j+N)} \end{bmatrix}, \quad \check{\mathbf{h}}_j = \begin{bmatrix} h_3^{(j)} \\ h_4^{(j+N)} \end{bmatrix}, \quad \mathbf{h}_j = \begin{bmatrix} h_2^{(j)} \\ h_3^{(j+N)} \\ h_4^{(j+2N)} \end{bmatrix}.
\end{aligned}$$

Note that the expressions (14) are linear in  $\{c_2^{(\nu)}, c_3^{(\nu+N)}, c_4^{(\nu+2N)} \mid \nu = N, \dots, 2N\}$ .

To proceed we need to define the *characteristic polynomial*

$$P(w, \lambda) = \det(\mathbb{L}(\lambda) - w\mathbb{I}). \quad (15)$$

It defines the spectral curve  $\mathcal{R}$ :

$$w^4 - I_2(\lambda)w^2 - I_3(\lambda)w - I_4(\lambda) = 0, \quad (16)$$

which is a curve of genus  $4N - 3$  in general. The spectral curve is common for integrable systems over orbits of both types:  $\mathcal{O}_f$  and  $\mathcal{O}_s$ . Restriction to an orbit is realized through the orbit equations (9) or (10), which fix some coefficients in (16). The rest of coefficients serve as Hamiltonians on an orbit and also remain constant during the evolution of a system.

Consider the spectral curve restricted to the orbit  $\mathcal{O}_f$ . Denoting its points by  $\{(\lambda_k, w_k)\}$  we write the following set of equations for  $6N$  Hamiltonians

$$\begin{aligned}
w_k^4 &= w_k^2 \left( h_2^{(0)} + h_2^{(1)} \lambda_k + \cdots h_2^{(N-1)} \lambda_k^{N-1} + c_2^{(N)} \lambda_k^N + c_2^{(N+1)} \lambda_k^{N+1} + \cdots + c_2^{(2N)} \lambda_k^{2N} \right) + \\
&+ w_k \left( h_3^{(0)} + h_3^{(1)} \lambda_k + \cdots h_3^{(2N-1)} \lambda_k^{2N-1} + c_3^{(2N)} \lambda_k^{2N} + c_3^{(2N+1)} \lambda_k^{2N+1} + \cdots + c_3^{(3N)} \lambda_k^{3N} \right) + \\
&+ \left( h_4^{(0)} + h_4^{(1)} \lambda_k + \cdots h_4^{(3N-1)} \lambda_k^{3N-1} + c_4^{(3N)} \lambda_k^{3N} + c_4^{(3N+1)} \lambda_k^{3N+1} + \cdots + c_4^{(4N)} \lambda_k^{4N} \right). \quad (17)
\end{aligned}$$

So we need  $6N$  points such that the system (17) is uniquely solved for the Hamiltonians. Rewrite the above system of equations in the matrix form

$$\begin{aligned}
W^- \mathbf{h} + W^+ \mathbf{c} &= \mathbf{w}, \\
W^- &= \begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{N-1} & W_1^0 & \lambda_1 W_1^0 & \cdots & \lambda_1^{N-1} W_1^0 & W_1 & \lambda_1 W_1 & \cdots & \lambda_1^{N-1} W_1 \\ 1 & \lambda_2 & \cdots & \lambda_2^{N-1} & W_2^0 & \lambda_2 W_2^0 & \cdots & \lambda_2^{N-1} W_2^0 & W_2 & \lambda_2 W_2 & \cdots & \lambda_2^{N-1} W_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \lambda_{6N} & \cdots & \lambda_{6N}^{N-1} & W_{6N}^0 & \lambda_{6N} W_{6N}^0 & \cdots & \lambda_{6N}^{N-1} W_{6N}^0 & W_{6N} & \lambda_{6N} W_{6N} & \cdots & \lambda_{6N}^{N-1} W_{6N} \end{bmatrix}, \\
W_k^0 &= [w_k \quad \lambda_k^N], \quad W_k = [w_k^2 \quad w_k \lambda_k^N \quad \lambda_k^{2N}]
\end{aligned}$$

$$W^+ = \begin{bmatrix} \lambda_1^N W_1 & \lambda_1^{N+1} W_1 & \dots & \lambda_1^{2N} W_1 \\ \lambda_2^N W_2 & \lambda_2^{N+1} W_2 & \dots & \lambda_2^{2N} W_2 \\ \vdots & \vdots & \dots & \vdots \\ \lambda_{6N}^N W_{6N} & \lambda_{6N}^{N+1} W_{6N} & \dots & \lambda_{6N}^{2N} W_{6N} \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} w_1^4 \\ w_2^4 \\ \vdots \\ w_{6N}^4 \end{bmatrix}.$$

Suppose that all pairs  $\{(\lambda_k, w_k) \mid k = 1, \dots, 6N\}$  are distinct points and  $W^-$  is nonsingular, then the Hamiltonians can be expressed by the formula

$$\mathbf{h} = -(W^-)^{-1} W^+ \mathbf{c} + (W^-)^{-1} \mathbf{w}. \quad (18)$$

On the orbit  $\mathcal{O}_f$  the formulas (14) and (18) define the same set of functions, moreover, both of them are linear in the parameters  $\{c_2^{(\nu)}, c_3^{(\nu+N)}, c_4^{(\nu+2N)} \mid \nu = N, \dots, 2N\}$  of the orbit. These parameters are independent, so one can equate the corresponding terms, that is

$$\begin{aligned} F^-(F^+)^{-1} &= -(W^-)^{-1} W^+, & \mathbf{a}^- - F^-(F^+)^{-1} \mathbf{a}^+ &= (W^-)^{-1} \mathbf{w} \Rightarrow \\ W^- F^- + W^+ F^+ &= 0, & W^- \mathbf{a}^- + W^+ \mathbf{a}^+ &= \mathbf{w}. \end{aligned} \quad (19)$$

The first matrix equation of (19) gives the following

$$\begin{bmatrix} B_{16}(\lambda_k) & B_{26}(\lambda_k) & B_{36}(\lambda_k) \\ B_{15}(\lambda_k) & B_{25}(\lambda_k) & B_{35}(\lambda_k) \\ B_{14}(\lambda_k) & B_{24}(\lambda_k) & B_{34}(\lambda_k) \end{bmatrix} \begin{bmatrix} w_k^2 \\ w_k \\ 1 \end{bmatrix} = \mathbf{0}. \quad (20)$$

where the entries are polynomials in  $\lambda_k$  with the coefficients defined by (13b), in general  $B_{36}$ ,  $B_{35}$ ,  $B_{34}$  are polynomials of degree  $3N$ ,  $B_{26}$ ,  $B_{25}$ ,  $B_{24}$  are polynomials of degree  $2N$ , and  $B_{16}(\lambda) = \gamma_6(\lambda)$ ,  $B_{15}(\lambda) = \gamma_5(\lambda)$ ,  $B_{14}(\lambda) = \gamma_4(\lambda)$ . We denote by  $B(\lambda_k)$  the matrix polynomial serving as a coefficient matrix of (20), namely:

$$B(\lambda) = B_{3N} \lambda^{3N} + \dots + B_1 \lambda + B_0.$$

The system of equations (20) is the main result of the proposed scheme. It allows to compute the set of points  $\{(\lambda_k, w_k)\}$  serving as variables of separation for Hamiltonian systems on a generic orbit of  $SU(4)$  loop group. In what follows we consider (20) as equations for  $(\lambda, w)$ :

$$B(\lambda) \begin{bmatrix} w^2 \\ w \\ 1 \end{bmatrix} = \mathbf{0}. \quad (21)$$

Nontrivial solutions for  $w$  exist if

$$\det B(\lambda) = 0, \quad (22)$$

which is an algebraic equation of degree  $6N$  if  $B_{3N}$  is nonsingular. Roots of  $\det B$  give the set of  $\lambda$ -variables forming a half of variables of separation, we suppose all  $\lambda_k$  are distinct. At every  $\lambda_k$  the L-matrix has 4 eigenvalues, one on every sheet of the spectral curve  $\mathcal{R}$ .

We solve (21) by the Gauss method:

$$\begin{aligned} & \begin{bmatrix} 0 & \begin{vmatrix} B_{16}(\lambda) & B_{26}(\lambda) \\ B_{14}(\lambda) & B_{24}(\lambda) \end{vmatrix} & \begin{vmatrix} B_{16}(\lambda) & B_{36}(\lambda) \\ B_{14}(\lambda) & B_{34}(\lambda) \end{vmatrix} \\ 0 & \begin{vmatrix} B_{15}(\lambda) & B_{25}(\lambda) \\ B_{14}(\lambda) & B_{24}(\lambda) \end{vmatrix} & \begin{vmatrix} B_{15}(\lambda) & B_{35}(\lambda) \\ B_{14}(\lambda) & B_{34}(\lambda) \end{vmatrix} \\ B_{14}(\lambda) & B_{24}(\lambda) & B_{34}(\lambda) \end{bmatrix} \begin{bmatrix} w^2 \\ w \\ 1 \end{bmatrix} = \mathbf{0}, \\ & w = -\frac{\begin{vmatrix} B_{15}(\lambda) & B_{35}(\lambda) \\ B_{14}(\lambda) & B_{34}(\lambda) \end{vmatrix}}{\begin{vmatrix} B_{15}(\lambda) & B_{25}(\lambda) \\ B_{14}(\lambda) & B_{24}(\lambda) \end{vmatrix}} = -\frac{\begin{vmatrix} B_{16}(\lambda) & B_{36}(\lambda) \\ B_{14}(\lambda) & B_{34}(\lambda) \end{vmatrix}}{\begin{vmatrix} B_{16}(\lambda) & B_{26}(\lambda) \\ B_{14}(\lambda) & B_{24}(\lambda) \end{vmatrix}} = -\frac{\begin{vmatrix} B_{16}(\lambda) & B_{36}(\lambda) \\ B_{15}(\lambda) & B_{35}(\lambda) \end{vmatrix}}{\begin{vmatrix} B_{16}(\lambda) & B_{26}(\lambda) \\ B_{15}(\lambda) & B_{25}(\lambda) \end{vmatrix}}. \end{aligned} \quad (23)$$

The last expression for  $w$  is obtained by using row 2 as leading. In this way we compute one  $w_k$  for each  $\lambda_k$ , and all the points  $(\lambda_k, w_k)$  are located on the same sheet of the spectral curve  $\mathcal{R}$  defined by (16).

The second matrix equation (19) gives

$$\begin{aligned} w_k^4 &= w_k^2 a_2(\lambda_k) + w_k a_3(\lambda_k) + a_4(\lambda_k), \quad \text{or} \\ [w_k + \alpha_3(\lambda_k)] [w_k^3 - \alpha_3(\lambda_k) w_k^2 - A_2(\lambda_k) w_k - A_3(\lambda_k)] &= 0, \end{aligned} \quad (24)$$

where  $a_2, a_3, a_4, A_2, A_3$  are polynomials with the coefficients defined by (13c). The equation (24) gives a simplification of the spectral curve equation (16) realized at every point of the set  $\{\lambda, w\} \equiv \{(\lambda_k, w_k) : k = 1, \dots, 6N\}$ . The variables  $\{\lambda, w\}$  give another parametrization of  $\mathcal{O}_f$ , and serve as variables of separation (see a proof in [3]).

#### 4.1 Connection to the known results

Recall some known results. In [4] one can find Sklyanin's Conjecture 1 asserting an existence of a polynomial  $\mathcal{B}$  whose roots serve as a half of variables of separation and a function  $\mathcal{A}$  giving the other half of variables of separation, that is

$$\mathcal{B}(\lambda_k) = 0, \quad w_k = \mathcal{A}(\lambda_k),$$

where  $\{(\lambda_k, w_k)\}$  are canonically conjugate:

$$\{\lambda_k, \lambda_l\} = 0, \quad \{\lambda_k, w_l\} = \delta_{kl}, \quad \{w_k, w_l\} = 0.$$

In [4] explicit expressions for the classical  $SL(3)$  magnetic chain are presented, and canonical conjugation is proven for this particular case. A fine proof of canonical conjugation for the loop group  $GL(r)$  of arbitrary rank  $r$  can be found in [3].

A thorough development of Sklyanin's idea is realized by Gekhtman in [5], where formulas for calculation of  $\mathcal{B}$  and  $\mathcal{A}$  are presented. Unfortunately, in [5] the result is given as a calculation technique without any explanation of grounds. Such situation provokes further investigation of the problem. One of explanation based on orbit method is presented in [1]. Its development for the  $SL(4)$  case is presented in this paper.

Regarding Conjecture 1 from Sklyanin's paper the polynomial  $\det \mathbf{B}$  is the polynomial  $\mathcal{B}$  whose roots serve as a half of variables of separation. And the expressions (23) serve as the function  $\mathcal{A}$ , compare them with the result from [5]. After Gekhtman one should take the last column without last entry of  $\mathbf{L}$ -matrix, we denote this vector by  $\xi$ , and the rest of  $\mathbf{L}$ -matrix without the last row, we denote this matrix by  $\mathbf{T}$ . To match this computation with the above expressions we use the transposed  $\mathbf{L}$ -matrix (without loss of generality):

$$\xi = \begin{bmatrix} \gamma_6(\lambda) \\ \gamma_5(\lambda) \\ \gamma_4(\lambda) \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} \alpha_1(\lambda) & \gamma_1(\lambda) & \gamma_3(\lambda) \\ \beta_1(\lambda) & \alpha_1(\lambda) - \alpha_1(\lambda) & \gamma_2(\lambda) \\ \beta_3(\lambda) & \beta_2(\lambda) & \alpha_3(\lambda) - \alpha_2(\lambda) \end{bmatrix}.$$

Then construct the square matrices:

$$\mathbf{S} = [\xi, \mathbf{T}\xi, \mathbf{T}^2\xi], \quad \mathbf{S}^{(1)} = [\mathbf{t}^{(1)}, \xi, \mathbf{T}\xi], \quad \mathbf{S}^{(2)} = [\mathbf{t}^{(2)}, \xi, \mathbf{T}\xi], \quad \mathbf{S}^{(3)} = [\mathbf{t}^{(3)}, \xi, \mathbf{T}\xi],$$

where  $\mathbf{t}^{(j)}$  are columns of  $\mathbf{T}$ . Then  $\mathcal{B}$  and  $\mathcal{A}$  are given by the formulas:

$$\mathcal{B}(\lambda) = \det \mathbf{S}, \quad \mathcal{A}(\lambda) = \frac{\det \mathbf{S}^{(1)}}{\det \check{\mathbf{S}}_1^3} = -\frac{\det \mathbf{S}^{(2)}}{\det \check{\mathbf{S}}_2^3} = \frac{\det \mathbf{S}^{(3)}}{\det \check{\mathbf{S}}_3^3}, \quad (25)$$

where  $\check{\mathbf{S}}_k^j$  is obtained from  $\mathbf{S}$  by elimination of row  $k$  and column  $j$ .

Expressions for  $\mathcal{B}$  and  $\mathcal{A}$  functions given by (23) and (25) coincide. The matrices  $\mathbf{B}$  and  $\mathbf{S}$  differ in a singular matrix, the same is true for the other corresponding matrices in the expressions:

$$\begin{aligned} \begin{vmatrix} B_{15}(\lambda) & B_{35}(\lambda) \\ B_{14}(\lambda) & B_{34}(\lambda) \end{vmatrix} &= -\det \mathbf{S}^{(1)}, & \begin{vmatrix} B_{16}(\lambda) & B_{36}(\lambda) \\ B_{14}(\lambda) & B_{34}(\lambda) \end{vmatrix} &= \det \mathbf{S}^{(2)}, & \begin{vmatrix} B_{16}(\lambda) & B_{36}(\lambda) \\ B_{15}(\lambda) & B_{35}(\lambda) \end{vmatrix} &= -\det \mathbf{S}^{(3)}, \\ \begin{vmatrix} B_{15}(\lambda) & B_{25}(\lambda) \\ B_{14}(\lambda) & B_{24}(\lambda) \end{vmatrix} &= \det \check{\mathbf{S}}_1^3, & \begin{vmatrix} B_{16}(\lambda) & B_{26}(\lambda) \\ B_{14}(\lambda) & B_{24}(\lambda) \end{vmatrix} &= \det \check{\mathbf{S}}_2^3, & \begin{vmatrix} B_{16}(\lambda) & B_{26}(\lambda) \\ B_{15}(\lambda) & B_{25}(\lambda) \end{vmatrix} &= \det \check{\mathbf{S}}_3^3. \end{aligned}$$



The problem of separation of variables on coadjoint orbits of the loop group  $GL(r)$  is also considered in [3, 6]. For further explanation we introduce the matrix  $\mathbf{N}(\lambda, w) \equiv \mathbf{L}(\lambda) - w\mathbb{I}$ , and denote by  $\tilde{\mathbf{N}}$  its adjoint matrix whose entries  $\tilde{N}_{ij}$  are cofactors corresponding to the entries  $N_{ji}$  of  $\mathbf{N}$ . Adams, Harnad and Hurtubise show that variables of separation (spectral Darboux coordinates) are zeros of  $\tilde{\mathbf{N}}(\lambda, w)\mathbf{v}_0$  with an arbitrary vector  $\mathbf{v}_0$  usually chosen as  $(1, 0, \dots, 0)^T$ . Applying this idea to the orbit  $\mathcal{O}_f$  in the loop algebra  $\mathfrak{sl}(4)$  we replace  $\tilde{\mathbf{N}}$  by its transpose and use another vector  $\mathbf{v}_0$

$$\tilde{\mathbf{N}}^t(\lambda, w) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \tilde{N}_{41}(\lambda, w) \\ \tilde{N}_{42}(\lambda, w) \\ \tilde{N}_{43}(\lambda, w) \\ \tilde{N}_{44}(\lambda, w) \end{pmatrix} = \mathbf{0}.$$

$$\begin{aligned} \tilde{N}_{41}(\lambda, w) &\equiv \begin{vmatrix} \gamma_1(\lambda) & \alpha_2(\lambda) - \alpha_1(\lambda) - w & \beta_2(\lambda) \\ \gamma_3(\lambda) & \gamma_2(\lambda) & \alpha_3(\lambda) - \alpha_2(\lambda) - w \\ \gamma_6(\lambda) & \gamma_5(\lambda) & \gamma_4(\lambda) \end{vmatrix} = w^2 B_{16}(\lambda) + w B_{26}(\lambda) + B_{36}, \\ \tilde{N}_{42}(\lambda, w) &\equiv - \begin{vmatrix} \alpha_1(\lambda) - w & \beta_1(\lambda) & \beta_3(\lambda) \\ \gamma_3(\lambda) & \gamma_2(\lambda) & \alpha_3(\lambda) - \alpha_2(\lambda) - w \\ \gamma_6(\lambda) & \gamma_5(\lambda) & \gamma_4(\lambda) \end{vmatrix} = w^2 B_{15}(\lambda) + w B_{25}(\lambda) + B_{35}, \\ \tilde{N}_{43}(\lambda, w) &\equiv - \begin{vmatrix} \alpha_1(\lambda) - w & \beta_1(\lambda) & \beta_3(\lambda) \\ \gamma_1(\lambda) & \alpha_2(\lambda) - \alpha_1(\lambda) - w & \beta_2(\lambda) \\ \gamma_6(\lambda) & \gamma_5(\lambda) & \gamma_4(\lambda) \end{vmatrix} = w^2 B_{14}(\lambda) + w B_{24}(\lambda) + B_{34}, \end{aligned}$$

One can see that (21) coincides with the following

$$\tilde{N}_{41}(\lambda, w) = 0, \quad \tilde{N}_{42}(\lambda, w) = 0, \quad \tilde{N}_{43}(\lambda, w) = 0. \quad (26)$$

The last equation is a simplification of the spectral curve equation, true only for the set  $\{(\lambda_k, w_k)\}$  satisfying both the other two equations. That is why the equation  $\tilde{N}_{44}(\lambda, w) = 0$ .

We see the obtained result are in good correspondence with the known ones. Moreover, the proposed orbit approach gives an obvious geometric explanation to the algorithm declared in [7], and can be easily extended to algebras of higher rank.

## 4.2 Mnemonic rule

Using a mnemonic rule we obtain

$$\begin{aligned} \begin{pmatrix} I_2(\lambda) \\ I_3(\lambda) \\ I_4(\lambda) \end{pmatrix} &= \mathbf{B}(\lambda) \begin{pmatrix} \beta_6(\lambda) \\ \beta_5(\lambda) \\ \beta_4(\lambda) \end{pmatrix} + \begin{pmatrix} A_2(\lambda) + \alpha_3^2(\lambda) \\ A_3(\lambda) + \alpha_3(\lambda)A_2(\lambda) \\ \alpha_3(\lambda)A_3(\lambda) \end{pmatrix} \\ \mathbf{B}^t(\lambda) &= \begin{bmatrix} \gamma_6 & \begin{vmatrix} \gamma_1 & \alpha_2 - \alpha_1 \\ \gamma_6 & \gamma_5 \end{vmatrix} + \begin{vmatrix} \gamma_3 & \alpha_3 - \alpha_2 \\ \gamma_6 & \gamma_4 \end{vmatrix} & \begin{vmatrix} \gamma_1 & \alpha_2 - \alpha_1 & \beta_2 \\ \gamma_3 & \gamma_2 & \alpha_3 - \alpha_2 \\ \gamma_6 & \gamma_5 & \gamma_4 \end{vmatrix} \\ \gamma_5 & - \begin{vmatrix} \alpha_1 & \beta_1 \\ \gamma_6 & \gamma_5 \end{vmatrix} + \begin{vmatrix} \gamma_2 & \alpha_3 - \alpha_2 \\ \gamma_5 & \gamma_4 \end{vmatrix} & - \begin{vmatrix} \alpha_1 & \beta_1 & \beta_3 \\ \gamma_3 & \gamma_2 & \alpha_3 - \alpha_2 \\ \gamma_6 & \gamma_5 & \gamma_4 \end{vmatrix} \\ \gamma_4 & - \begin{vmatrix} \alpha_1 & \beta_3 \\ \gamma_6 & \gamma_4 \end{vmatrix} - \begin{vmatrix} \alpha_2 - \alpha_1 & \beta_2 \\ \gamma_5 & \gamma_4 \end{vmatrix} & \begin{vmatrix} \alpha_1 & \beta_1 & \beta_3 \\ \gamma_1 & \alpha_2 - \alpha_1 & \beta_2 \\ \gamma_6 & \gamma_5 & \gamma_4 \end{vmatrix} \end{bmatrix} (\lambda), \\ A_2(\lambda) &= \alpha_1^2 + \alpha_2^2 - \alpha_1\alpha_2 - \alpha_2\alpha_3 + \beta_1\gamma_1 + \beta_2\gamma_2 + \beta_3\gamma_3 = \\ &= - \begin{vmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \alpha_2 - \alpha_1 \end{vmatrix} - \begin{vmatrix} \alpha_2 - \alpha_1 & \beta_2 \\ \gamma_2 & \alpha_3 - \alpha_2 \end{vmatrix} - \begin{vmatrix} \alpha_1 & \beta_3 \\ \gamma_3 & \alpha_3 - \alpha_2 \end{vmatrix}. \\ A_3(\lambda) &= \alpha_2\alpha_1^2 - \alpha_1\alpha_2^2 + \alpha_1\alpha_2\alpha_3 - \alpha_3\alpha_1^2 - \beta_1\gamma_1(\alpha_3 - \alpha_2) - \beta_2\gamma_2\alpha_1 - \beta_3\gamma_3(\alpha_2 - \alpha_1) + \\ &+ \beta_3\gamma_1\gamma_2 + \beta_1\beta_2\gamma_3 = \begin{vmatrix} \alpha_1 & \beta_1 & \beta_3 \\ \gamma_1 & \alpha_2 - \alpha_1 & \beta_2 \\ \gamma_3 & \gamma_2 & \alpha_3 - \alpha_2 \end{vmatrix}. \end{aligned}$$

We are looking for special points  $\{\lambda_k\}$  where the matrix  $\mathbf{B}$  is singular:  $\mathbf{B}(\lambda_k) = 0$ . By imposing the conditions  $\mathbf{B}(\lambda_k)\boldsymbol{\beta}(\lambda_k) = \mathbf{0}$  we come to the reduced equation of the spectral curve

$$(w + \alpha_3(\lambda_k))(w^3 - \alpha_3(\lambda_k)w^2 - A_2(\lambda_k)w - A_3(\lambda_k)) = 0.$$

## 5 Separation of variables Theorems

Summarizing the above computation we formulate the following

**Separation of variables theorem 1.** *Suppose the orbit  $\mathcal{O}_f$  is parameterized by the variables  $\{\alpha_1^{(m)}, \alpha_2^{(m)}, \alpha_3^{(m)}, \beta_1^{(m)}, \beta_2^{(m)}, \beta_3^{(m)}, \gamma_1^{(m)}, \gamma_2^{(m)}, \gamma_3^{(m)}, \gamma_4^{(m)}, \gamma_5^{(m)}, \gamma_6^{(m)} : m=0, \dots, N-1\}$  as above. Then the new variables  $\{(\lambda_k, w_k) : k=1, \dots, 6N\}$  defined by the formulas*

$$\mathcal{B}(\lambda_k) = 0, \quad w_k = \mathcal{A}(\lambda_k), \quad (27)$$

where  $\mathcal{B}$  is the polynomial of degree  $6N$  and  $\mathcal{A}$  is the algebraic function such that

$$\mathcal{B}(\lambda) = \det B(\lambda), \quad (28a)$$

$$\mathcal{A}(\lambda) = -\frac{\begin{vmatrix} B_{15}(\lambda) & B_{35}(\lambda) \\ B_{14}(\lambda) & B_{34}(\lambda) \end{vmatrix}}{\begin{vmatrix} B_{15}(\lambda) & B_{25}(\lambda) \\ B_{14}(\lambda) & B_{24}(\lambda) \end{vmatrix}} \quad \text{or} \quad -\frac{\begin{vmatrix} B_{16}(\lambda) & B_{36}(\lambda) \\ B_{14}(\lambda) & B_{34}(\lambda) \end{vmatrix}}{\begin{vmatrix} B_{16}(\lambda) & B_{26}(\lambda) \\ B_{14}(\lambda) & B_{24}(\lambda) \end{vmatrix}} \quad \text{or} \quad -\frac{\begin{vmatrix} B_{16}(\lambda) & B_{36}(\lambda) \\ B_{15}(\lambda) & B_{35}(\lambda) \end{vmatrix}}{\begin{vmatrix} B_{16}(\lambda) & B_{26}(\lambda) \\ B_{15}(\lambda) & B_{25}(\lambda) \end{vmatrix}}. \quad (28b)$$

have the following properties:

- (i) a pair  $(\lambda_k, w_k)$  is a root of the characteristic polynomial (15);
- (ii) a pair  $(\lambda_k, w_k)$  is canonically conjugate with respect to the first Lie-Poisson bracket (2):

$$\{\lambda_k, \lambda_l\}_f = 0, \quad \{\lambda_k, w_l\}_f = \delta_{kl}, \quad \{w_k, w_l\}_f = 0; \quad (29)$$

- (iii) the corresponding Liouville 1-form is

$$\Omega_f = \sum_k w_k d\lambda_k.$$

*Proof.* (i) The characteristic polynomial  $P$  defined by (15) has  $\mathcal{B}(\lambda_k)$  as a factor at every point  $(\lambda_k, w_k)$ , that is proven by explicit calculation. All the expressions (28b) for  $\mathcal{A}$  coincide at all zeros  $\{\lambda_k\}$  of  $\mathcal{B}$ , that is they give the same eigenvalue of the L-matrix (1). This proves the assertion (i).

(ii) To prove this assertion we use the Conjugate variable lemma 1 proven in [1] and recalled here, and  $\mathcal{A}$ - $\mathcal{B}$  bracket lemma 1 stated below.

**Conjugate variable lemma 1.** *If  $\mathcal{B}$  and  $\mathcal{A}$  satisfy the following identities with respect to the first Lie-Poisson bracket (11)*

$$\{\mathcal{B}(u), \mathcal{B}(v)\}_f = 0, \quad \{\mathcal{A}(u), \mathcal{A}(v)\}_f = 0, \quad \{\mathcal{A}(u), \mathcal{B}(v)\}_f = \frac{f(u, v)\mathcal{B}(u) - \mathcal{B}(v)}{u - v},$$

where  $f$  is an arbitrary function such that  $\lim_{v \rightarrow u} f(u, v) = 1$ , then the variables  $\{(\lambda_k, w_k) : k=1, \dots, 6N\}$  defined by

$$\mathcal{B}(\lambda_k) = 0, \quad w_k = \mathcal{A}(\lambda_k)$$

are canonically conjugate with respect to  $\{\cdot, \cdot\}_f$ :

$$\{\lambda_k, \lambda_l\}_f = 0, \quad \{\lambda_k, w_l\}_f = \delta_{kl}, \quad \{w_k, w_l\}_f = 0.$$

**$\mathcal{A}$ - $\mathcal{B}$  bracket lemma 1..** *For  $\mathcal{B}$  and  $\mathcal{A}$  defined by (28) the following identities are true with respect to the first Lie-Poisson bracket (11):*

$$\{\mathcal{B}(u), \mathcal{B}(v)\}_f = 0, \quad \{\mathcal{A}(u), \mathcal{A}(v)\}_f = 0, \quad \{\mathcal{A}(u), \mathcal{B}(v)\}_f = \frac{f(u, v)\mathcal{B}(u) - \mathcal{B}(v)}{u - v},$$

where  $f(u, v) = \dots$  for (28b).

(iii) The Liouville 1-form on  $\mathcal{O}_f$  is implied by (29):

$$\Omega_f = \sum_k w_k d\lambda_k.$$

By fixing values of the Hamiltonians  $h_2^{(0)}, h_2^{(1)}, \dots, h_2^{(N-1)}, h_3^{(0)}, h_3^{(1)}, \dots, h_3^{(2N-1)}, h_4^{(0)}, h_4^{(1)}, \dots, h_4^{(3N-1)}$  we obtain a Liouville torus. On the torus every variable  $w_k$  becomes an algebraic function of the corresponding conjugate variable  $\lambda_k$  due to (17), and the form  $\Omega_f$  becomes a sum of meromorphic differentials on the Riemann surface  $P(w, \lambda) = 0$ .

This completes the proof of Separation of variables theorem 1.  $\square$

Given a set of pairs  $\{(\lambda_k, w_k) : k = 1, \dots, 6N\}$  we are able to compute the dynamic variables satisfying the equations (27). Thus, one defines a homomorphism

$$\mathbb{C}^{6N} \rightarrow \mathcal{O}_f \quad (30)$$

that maps  $\{(\lambda_k, w_k)\}$  to a point of  $\mathcal{O}_f$ . When all the Hamiltonians are fixed the homomorphism (30) turns into the map from the symmetrized product of  $6N$  Riemann surfaces  $\mathcal{R}$  defined by (17) to a Liouville torus:

$$\text{Sym}\{\mathcal{R} \times \mathcal{R} \times \dots \times \mathcal{R}\} \mapsto T^{6N}.$$

### Separation of variables on $\mathcal{O}_s$

Separation of variables on  $\mathcal{O}_s$  is also realized through restriction to the orbit, see [1] for details. One obtains the same matrix equation (20) producing the same expressions for  $\mathcal{A}$  and  $\mathcal{B}$ . It means the same points on the spectral curve (16) serve as variables of separation.

**Separation of variables theorem 2.** *Suppose the orbit  $\mathcal{O}_s$  is parameterized by the variables  $\{\alpha_1^{(m)}, \alpha_2^{(m)}, \alpha_3^{(m)}, \beta_1^{(m)}, \beta_2^{(m)}, \beta_3^{(m)}, \gamma_1^{(m)}, \gamma_2^{(m)}, \gamma_3^{(m)}, \gamma_4^{(m)}, \gamma_5^{(m)}, \gamma_6^{(m)} : m = 0, \dots, N-1\}$  as above. Then the new variables  $\{(\lambda_k, w_k) : k = 1, \dots, 6N\}$  defined by the formulas*

$$\mathcal{B}(\lambda_k) = 0, \quad w_k = \mathcal{A}(\lambda_k), \quad (31)$$

where  $\mathcal{B}$  is the polynomial of degree  $6N$  and  $\mathcal{A}$  is the algebraic function such that

$$\mathcal{B}(\lambda) = \det B(\lambda), \quad (32a)$$

$$\mathcal{A}(\lambda) = -\frac{\begin{vmatrix} B_{15}(\lambda) & B_{35}(\lambda) \\ B_{14}(\lambda) & B_{34}(\lambda) \end{vmatrix}}{\begin{vmatrix} B_{15}(\lambda) & B_{25}(\lambda) \\ B_{14}(\lambda) & B_{24}(\lambda) \end{vmatrix}} \quad \text{or} \quad -\frac{\begin{vmatrix} B_{16}(\lambda) & B_{36}(\lambda) \\ B_{14}(\lambda) & B_{34}(\lambda) \end{vmatrix}}{\begin{vmatrix} B_{16}(\lambda) & B_{26}(\lambda) \\ B_{14}(\lambda) & B_{24}(\lambda) \end{vmatrix}} \quad \text{or} \quad -\frac{\begin{vmatrix} B_{16}(\lambda) & B_{36}(\lambda) \\ B_{15}(\lambda) & B_{35}(\lambda) \end{vmatrix}}{\begin{vmatrix} B_{16}(\lambda) & B_{26}(\lambda) \\ B_{15}(\lambda) & B_{25}(\lambda) \end{vmatrix}}. \quad (32b)$$

- (i) a pair  $(\lambda_k, w_k)$  is a root of the characteristic polynomial (15).
- (ii) a pair  $(\lambda_k, w_k)$  is quasi-canonically conjugate with respect to the second Lie-Poisson bracket (3):

$$\{\lambda_k, \lambda_l\}_s = 0 \quad \{\lambda_k, w_l\}_s = -\lambda_k^N \delta_{kl}, \quad \{w_k, w_l\}_s = 0; \quad (33)$$

- (iii) the corresponding Liouville 1-form is

$$\Omega_s = -\sum_k \lambda_k^{-N} w_k d\lambda_k.$$

*Proof.* (i) The proof repeats one for the Separation of variables theorem 1.

(ii) The assertion follows from the lemmas below.

**Conjugate variable lemma 2.** *If  $\mathcal{B}$  and  $\mathcal{A}$  satisfy the following identities with respect to the second Lie-Poisson bracket (3)*

$$\{\mathcal{B}(u), \mathcal{B}(v)\}_s = 0, \quad \{\mathcal{A}(u), \mathcal{A}(v)\}_s = 0,$$

$$\{\mathcal{A}(u), \mathcal{B}(v)\}_s = \frac{u^N \mathcal{B}(v) - v^N \mathcal{B}(u) f(u, v)}{u - v},$$

where  $f$  is an arbitrary function such that  $\lim_{v \rightarrow u} f(u, v) = 1$ , then the variables  $\{(\lambda_k, w_k)\}$  defined by

$$\mathcal{B}(\lambda_k) = 0, \quad w_k = \mathcal{A}(\lambda_k)$$

are quasi-canonically conjugate with respect to  $\{\cdot, \cdot\}_s$ :

$$\{\lambda_k, \lambda_l\}_s = 0, \quad \{\lambda_k, w_l\}_s = -\lambda_k^N \delta_{kl}, \quad \{w_k, w_l\}_s = 0.$$

**A-B bracket lemma 2..** For  $\mathcal{B}$  and  $\mathcal{A}$  defined by (28) the following identities are true with respect to the second Lie-Poisson bracket (3)

$$\begin{aligned} \{\mathcal{B}(u), \mathcal{B}(v)\}_s &= 0, \quad \{\mathcal{A}(u), \mathcal{A}(v)\}_s = 0, \\ \{\mathcal{A}(u), \mathcal{B}(v)\}_s &= \frac{u^N \mathcal{B}(v) - v^N \mathcal{B}(u) f(u, v)}{u - v}, \end{aligned}$$

where  $f(u, v) = \dots$  for (28b).

(iii) The Liouville 1-form on  $\mathcal{O}_s$  is implied by (33):

$$\Omega_s = - \sum_k \lambda^{-N} w_k d\lambda_k.$$

Reduction to a Liouville torus is realized by fixing values of the Hamiltonians  $h_2^{(N)}, h_2^{(N+1)}, \dots, h_2^{(2N-1)}, h_3^{(N)}, h_3^{(N+1)}, \dots, h_3^{(3N-1)}, h_4^{(N)}, h_4^{(N+1)}, \dots, h_4^{(4N-1)}$ . On the torus every  $w_k$  is an algebraic function of the conjugate variable  $\lambda_k$ . After this reduction the form  $\Omega_s$  becomes a sum of meromorphic differentials on the Riemann surface  $P(w, \lambda) = 0$ .

This completes the proof of Separation of variables theorem 2.  $\square$

Above we suppose the matrix polynomial  $\mathbf{B}$  has the maximal degree  $6N$ . If not one should apply to  $\mathbf{L}$ -matrix a proper similarity transformation that makes  $\mathbf{B}$  of maximal degree.

## 6 Conclusion and Discussion

Here a brief summary of the orbit approach is given. Recall that an integrable system is constructed on a coadjoint orbit in the loop Lie algebra  $\widetilde{\mathfrak{g}}$ , it means smooth functions on the dual space to  $\widetilde{\mathfrak{g}}$  serves as a phase space of integrable system. We use the Cartan-Weyl basis in the Lie algebra and restrict the phase space to an orbit through eliminating a subset of dynamic variables corresponding to nilpotent commuting basis elements. The rest of dynamic variables corresponding to basis elements give a parametrization of the orbit. Another parametrization of the orbit is given by points of the spectral curve  $\det(\mathbf{L}(\lambda) - w \mathbb{I}) = 0$ , where  $\mathbf{L}$  is the Lax matrix of the system. Thus, we obtain a map between the dynamic and the spectral variables, it is possible to make this map biunique. The spectral variables are proven to be variables of separation.

The orbit approach allows an easy extension to generic orbits in  $\mathfrak{sl}(n)$  loop algebras. At the same time expressions for the functions  $\mathcal{A}$  and  $\mathcal{B}$  giving variables of separation acquire a reasonable meaning: they are implied by the procedure of restriction to an orbit, and a simple mnemonic rule allows to write them immediately.

## References

- [1] Bernatska J., Holod P. arXiv:1312.1975.
- [2] Adler M. and van Moerbeke P., Adv. Math. **38**, (1980) 318–379.
- [3] Adams M. R., Harnad J., Hurtubise J., Commun. Math. Phys. **155**, (1993) 385–413.

- [4] Sklyanin E. K., Commun. Math. Phys. **150**, (1992) 181–191.
- [5] Gekhtman M. I., Commun. Math. Phys. **167**, (1995) 593–605.
- [6] Adams M. R., Harnad J., Hurtubise J., Lett. Math. Phys. **40**, (1997) 41–57.
- [7] Sklyanin E. K., Progr. Theor. Phys. Suppl. **118**, (1995) 35–60.